# Collatz Mapping on \mathbb{Z}_{10}

Benyamin Khanzadeh Holasou



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BENYAMIN KHANZADEH H.

# 1 Introduction

The *Collatz conjecture*, also known as the "$3x+1$" problem, is a conjecture in mathematics that concerns sequences of integer numbers. These sequences start with an arbitrary positive integer, and so each term is obtained from the previous one as follows: if the previous term is even, the next term is one half of this one, and if the previous term is odd, the next term is 3 times the previous one plus 1. The conjecture says that no matter what is the starting value, the sequence will always reach 1. [1]

More specifically, consider the *Collatz function* as the map $\mathrm{Col} : \mathbb{N} \to \mathbb{N}$, defined by:

$$\mathrm{Col}(x) = \begin{cases} \frac{x}{2} & \text{if } x \equiv 0 \mod 2 \\ 3x+1 & \text{if } x \equiv 1 \mod 2 \end{cases}$$

Thus, the Collatz conjecture says that, for every $x \in \mathbb{N}$, there is a positive integer $k$ such that $\mathrm{Col}^k(x) = 1$. [1]

The conjecture is named after Lothar Collatz, who introduced the idea in 1937, two years after receiving his doctorate [1]. However, it is also known as the *Ulam conjecture* (after Stanisław Ulam), the *Kakutani's problem* (after Shizuo Kakutani), the *Thwaites conjecture* (after Sir Bryan Thwaites), the *Hasse's algorithm* (after Helmut Hasse), or the *Syracuse problem* [2, 3].

The sequences of numbers involved here are also referred to as the *hailstone sequences* or the *hailstone numbers*, because the values are usually subject to multiple descents and ascents like hailstones in a cloud [**?**, **?**] or as wondrous numbers [**?**].

Paul Erdős said about the Collatz conjecture: "mathematics may not be ready for such problems" [**?**]. He also offered US$500 for its solution [**?**]. On the other hand, Jeffrey Lagarias stated in 2010 that the Collatz conjecture "is an extraordinarily difficult problem, completely out of reach of present day mathematics" [**?**].

# 2 Main conjecture

## 2.1 Closed loops

A *closed loop* or simply a *cycle* is a finite sequence $a = (a_0, \ldots, a_k)$ of positive integers, such that $a_k = a_0$ and $a_i = \mathrm{Col}^i(a_{i-1})$ for all $i \in \{1, \ldots, k\}$. Note that every closed loop $a$ as above generates an infinite family of cycles $a^m$ $(m \geq 1)$, by concatenating $m-1$ copies of $(a_1, \ldots, a_k)$ to the right of $a$, that is

$$a^m = (a_0, \underbrace{a_1, \ldots, a_k}_{(m-1)-\text{copies}}).$$

Hence, if $a$ is a closed loop as above, then $\mathrm{Col}^{mk}(a_0) = a_0$ for all $m \geq 1$.

One of the most important problems about this conjecture is the presence of the closed loop $(1, 4, 2, 1)$, because it creates complexity. Indeed, if $\mathrm{Col}^k(x) = 1$, the existence of the mentioned closed loop implies that

$$\mathrm{Col}^{3m}(\mathrm{Col}^k(x)) = \mathrm{Col}^{3m+k}(x) = 1 \quad \text{for all} \quad m \geq 1.$$

To avoid the contradiction given by the closed loop $(1, 4, 2, 1)$, we will replace the Collatz function by the map $\mathrm{Col}_* : \mathbb{N} \to \mathbb{N}$ defined as follows:

$$\mathrm{Col}_*(x) = \begin{cases} \frac{x}{2} & \text{if } x \equiv 0 \mod 2 \\ 3x+1 & \text{if } x \equiv 1 \mod 2 \text{ and } x > 1 \\ 1 & \text{if } x = 1 \end{cases}$$

For $t \in \mathbb{N}$, we have

$$\mathrm{Col}(2t+1) = 3(2t+1) + 1 = 6t + 4 \equiv 4 \mod 6, \qquad \mathrm{Col}(2t) = \frac{2t}{2} = t.$$

Notice that, for $x \in \mathbb{N}$, $\mathrm{Col}^{-1}(\{x\})$ is formed by the element $2x$, and additionally, by the element $\frac{x-1}{3}$ only if $x \equiv 4 \mod 6$.

# 3 Classifications and fields

For each integer $m$ such that $0 < m < 10$, denote by $[m]$ the class of it module ten, that is

$$\begin{aligned} [0] &= \{0, 10, 20, \ldots\}, \\ [1] &= \{1, 11, 21, \ldots\}, \\ &\cdots \\ [9] &= \{9, 19, 29, \ldots\}. \end{aligned}$$

Thus,

$$\mathbb{Z}_{10} = \{[0], \ldots, [9]\} \quad \text{and} \quad \mathbb{N} = [0] \sqcup \cdots \sqcup [9].$$

As every natural number $n$ can be written as

$$n = \sum_{k=2}^{n} a_k 10^{k-1}, \qquad 0 < a_2, \ldots, a_n < 10.$$

As every natural number can be written as $\sum_{k=2}^{n} a_k 10^{k-1}$ with $0 < a_k < 10$, then

$$[0] = \left\{ \sum_{k=2}^{n} \mid \right\}.$$

We start by defining a set of natural numbers $\mathbb{N} = \{1, 2, 3, 4, 5, \cdots\}$ and write it as a community of classes, and with this definition we will arrive at a simpler set and express it as a field of $\mathbb{Z}$ expression. ($\mathbb{N} = \mathbb{Z}^{+}$) As mentioned in the introduction (inverse function part), each part of Collatz conjectures contains a series of distinct properties, but in this section we will classify these properties as simply as possible and make the algebraic properties we want from them.

We will first define a set of natural numbers and then extend the Collatz function through the properties we have obtained in this classification.

**Definition 3.1** $\mathbb{N} = \bigcup_{0 \le i \le 9} [i]_{10}^{+}$ , $[i]_{10}^{+} = \{x : x = \sum_{j=2}^{n} a_j 10^{j-1} + i, x \geq 1, 0 \leq \forall a_j \leq 9\} = \{10k + i : k \geq 0\}$ *( and if* $i > 0$ *we will have :* $[i]_{10}^{+} = \{i\} \cup [0]_{10}$*).*

Which is quite obvious because:

$$\mathbb{N} = \{10, 20, 30, \cdots\} \cup \{1, 11, 21, 31, \cdots\} \cup \cdots \{9, 19, 29, \cdots\}$$

Now according to the above definition and definition $\mathbb{Z}_{10}^{+}$ we will have:

$$\mathbb{N} = \mathbb{Z}_{10}^{+}$$

Now, according to the definitions we have had so far, we want to study the properties of each of these classes, and by studying their properties, we will reach more general properties in the $\mathbb{Z}_{10}$ set, and with the help of these properties, we can generalize them to the set of $\mathbb{N}$ numbers.

### 3.1 Properties of classes

Now for searching to the properties that classes have we use these lemmas:

**Lemma 3.1.1** $\forall x \in [2k+1]^{+}_{10} \Rightarrow \mathrm{Col}(x) \in [6k+4]^{+}_{10}$

**Proof** To proof it we can say that:

$$x \in [2k+1]^{+}_{10} \Rightarrow x = \sum_{j=2}^{n} a_j 10^{j-1} + 2k+1, 0 \leq \forall a_j \leq 9$$

So we will have:
$\mathrm{Col}(x) = 3x+1 = 3(\sum_{j=2}^{n} a_j 10^{j-1} + 2k+1) + 1 = 3\sum_{j=2}^{n} a_j 10^{j-1} + 6k+4 \equiv 6k+4 \pmod{10} \Rightarrow \mathrm{Col}(x) \in [6k+4]^{+}_{10}$ □

**Lemma 3.1.2** $\forall x \in [2k]^{+}_{10} \Rightarrow \mathrm{Col}(x) \in [k]^{+}_{10} \cup [k+5]^{+}_{10}$

**Proof** To proof it we can do the same thing that we done in Lemma 3.1.1.:

$$x \in [2k]^{+}_{10} \Rightarrow x = \sum_{j=2}^{n} a_j 10^{j-1} + 2k, 0 \leq \forall a_j \leq 9$$

$\mathrm{Col}(x) = \frac{x}{2} = \sum_{j=2}^{n} \frac{a_j}{2} 10^{j-1} + k = \sum_{j=3}^{n} \frac{a_j}{2} 10^{j-1} + 5a_2 + k$
i) $a_2 = 2k_1$ then we will get: $\sum_{j=3}^{n} \frac{a_j}{2} 10^{j-1} + 10k_1 + k = 10k' + k \equiv k \pmod{10} \Rightarrow \mathrm{Col}(x) \in [k]^{+}_{10}$
ii) $a_2 = 2k_1 + 1$ then we will get: $\sum_{j=3}^{n} \frac{a_j}{2} 10^{j-1} + 10k_1 + 5 + k \equiv 5 + k \pmod{10} \Rightarrow \mathrm{Col}(x) \in [k+5]^{+}_{10}$ □

**Result 3.1.1** *because* $2k \in \{0,2,4,6,8\} \Rightarrow 0 \leq k \leq 4 \Rightarrow 5 \leq k+5 \leq 9$

Following result is important because we will use it in the future (for drawing graph):

**Result 3.1.2** *Each element on* $\mathbb{Z}_{10}$ *will have these properties:*
*Properties 0.* $x \in [0]^{+}_{10} \Rightarrow \mathrm{Col}(x) \in [5]^{+}_{10} \cup [0]^{+}_{10}$
*Properties 1.* $x \in [1]^{+}_{10} \Rightarrow \mathrm{Col}(x) \in [4]^{+}_{10}$
*Properties 2.* $x \in [2]^{+}_{10} \Rightarrow \mathrm{Col}(x) \in [1]^{+}_{10} \cup [6]^{+}_{10}$
*Properties 3.* $x \in [3]^{+}_{10} \Rightarrow \mathrm{Col}(x) \in [0]^{+}_{10}$
*Properties 4.* $x \in [4]^{+}_{10} \Rightarrow \mathrm{Col}(x) \in [2]^{+}_{10} \cup [7]^{+}_{10}$
*Properties 5.* $x \in [5]^{+}_{10} \Rightarrow \mathrm{Col}(x) \in [6]^{+}_{10}$
*Properties 6.* $x \in [6]^{+}_{10} \Rightarrow \mathrm{Col}(x) \in [8]^{+}_{10} \cup [3]^{+}_{10}$
*Properties 7.* $x \in [7]^{+}_{10} \Rightarrow \mathrm{Col}(x) \in [2]^{+}_{10}$
*Properties 8.* $x \in [8]^{+}_{10} \Rightarrow \mathrm{Col}(x) \in [4]^{+}_{10} \cup [9]^{+}_{10}$
*Properties 9.* $x \in [9]^{+}_{10} \Rightarrow \mathrm{Col}(x) \in [8]^{+}_{10}$

# 4 Graph drawing

In this part of the article, we intend to use the properties that we expressed and proved in the previous sections, and draw a graph that expresses all its properties.
To draw a given graph $G(V,E)$, we need two elements of its vertices and edges, which we will express all of them in the future.
The graph that we will draw in this section is a complete distribution to express the Collatz function in the set $\text{Col} : \mathbb{N} \to \mathbb{N} \Rightarrow \text{Col} : \mathbb{Z}_{10}^{+} \to \mathbb{Z}_{10}^{+}$ and we will prove later that we can extend the Collatz conjecture based on this graph and arrive at a proposition equivalent to the Collatz conjecture.

## 4.1 Graph properties

In this part of the article, we want to express the properties of a graph so that we can draw a given graph.
Given that the given graph is supposed to express the most basic properties of numbers in the Collatz system, and given that in the previous section we proved that the most basic properties of Collatz can be summarized in $\mathbb{Z}_{10}$ the vertices of the graph G can be written as follows :

$$V = \{i | i \in \mathbb{Z}_{10}^{+}\} \Rightarrow V = \{0,1,2,3,4,5,6,7,8,9\}$$

So it can be understood that the hypothetical graph is made up of 10 vertices, all in the field $\mathbb{Z}_{10}$.

Now, considering that we know in this field how classes are defined and how they are connected to each other, the edges of the assumed graph can be expressed as follows:

$$E = \{\vec{ij} : f(i) \equiv j \pmod{10}, \forall i,j \in \mathbb{Z}_{10}^{+}\}$$
(using Result 2.1.1)

Now, according to the above definitions, we can draw the assumed graph, but for a simpler drawing, we need the following two lemma:

**Lemma 4.1.1** $deg^{\pm}(x) = 1 \iff x \in \mathbb{O}$.

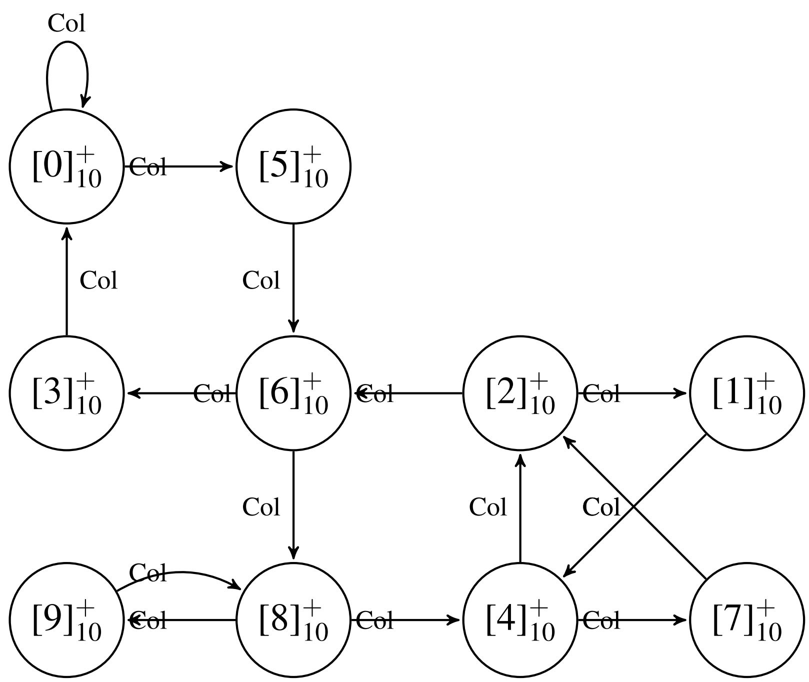
*Proof*. To proof it we just need to use result 3.1.2, lemma 3.1.1 and lemma 3.1.2 .■

**Lemma 4.1.2** $deg^{\pm}(x) = 2 \iff x \in \mathbb{E}$

*Proof*. To proof it we just need to use result 3.1.2, lemma 3.1.1 and lemma 3.1.2 .■

### 4.2 Final result

By using Lemma 3.1.1, 3.1.2, 4.1.1, 4.1.2 and $V$ , $E$ definiton we will have:

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## 5 Generalization of the Collatz conjecture

In this section, according to the graph that we drew in the previous section and the theorems and lemmas that we have dealt with so far, we want to change the form of Collatz's conjecture as follows:

**Conjecture 1** *All natural numbers in function* Col *will reach 1 if and only if:*

$$\begin{cases} i)\ \lim\limits_{k\to\infty} \mathrm{Col}^k(x) \in \{1,4,2\} \vee \lim\limits_{k\to k_1} \mathrm{Col}^k(x) \neq x \\ ii)\ \lim\limits_{k\to\infty} \mathrm{Col}^k(x) \neq \infty \end{cases} \qquad \forall x,k,k_1 \in \mathbb{N}$$

Now we can claim that this conjecture is equivalent to the Collatz conjecture, because if a number does not reach itself and also does not diverge infinitely, then that number will necessarily shrink to 1. In fact, we can say that the graph we have drawn shows that If there is a loop, then what does that loop look like, and if a proof can be given for this conjecture, this graph can be used.

## References

[1] J.C. Lagarias. The $3x + 1$ problem: An annotated bibliography (1963-1999). Sorted by author. URL: https://arxiv.org/abs/math/0309224.

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benjaminkhanzade319@gmail.com