An Algorithm for Extending Menger-Type Fractal Structures

Manuel Diaz Regueiro and Luis Diaz Allegue

April 20, 2023

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Manuel Diaz Regueiro[1] and Luis Diaz Allegue[2]

[1]Retired Math Teacher, Spain; mdregueiro2@gmail.com
[2]ANFACO-CECOPESCA, Department of Circular Economy, Spain; ldiaz@anfaco.es

Abstract

In this paper, the potential to generate a wide range of designs using an algorithm that extends Menger-type fractal constructions. The authors describe a method that can be used to build a Menger-type figure from which the initial "atoms", the rules and distances can be changed to create many interesting figures. Furthermore, this method can be applied in a new variation called Menger-Diaz. The Menger-Diaz variation is created by modifying the way initial "atoms" are removed at each level, resulting in a shape with unique properties and characteristics that are demonstrated through several examples. Therefore, this method can potentially lead to potential applications of these designs in fields such as architecture, engineering, and computer science, as well as the potential for using these designs to create artistic works.

Introduction

In the world of ideas there are many wonders awaiting us. Sierpinski- or Menger-like designs are a well-known and popular class of experimental designs that were first proposed by the mathematician Karl Menger in 1926 [1]. Menger Sponge (or cube) is a prime example of a fractal, a type of geometric shape that exhibits self-similarity at different scales [2].

The Menger Sponge has several applications in fields such as computer graphics, nanomaterials and data visualization and has been studied in fields such as mathematics, physics, and engineering, due to its unique properties and the insights it can provide into the behavior of complex systems [2]–[4].

In particular, the Menger Sponge is a three-dimensional fractal shape formed a cube in which each face is divided into 9 initial “atoms” and subsequently iteratively removing the central “atom”. This process can be repeated indefinitely, resulting in a shape with an infinite number of faces and an infinitely complex surface. The number of times the fractal creation process has been applied are normally called levels. As more levels are added, the fractal becomes more complex. However, as levels increase, the complexity of the fractal also increases, and it takes more time and resources to compute it.

In addition to the possibility of changing the first "atom" and the level of the Merger figures, we can also change their distances. In summary, there are two types of distance that are relevant: fixed distances and variable distances. Fixed distances are established at the beginning of the construction process and do not change as the level increases. These distances are typically related to the size of the first "atom". The fixed distances in Menger figures are important because they determine the overall size and shape of the figure, as well as the relative positions of the small cubes within it. Variable distances, on the other hand, are distances that change as the recursion level increases. In the case of the Menger Sponge, these distances are related to the number of "atoms" that lie between any two points within the figure. As the level increases, the number of "atoms" between any two points will also increase, resulting in a larger variable distance between those points.

Overall, both fixed- and variable-distances play important roles in the structure and properties of Menger figures. Nonetheless, in this paper, when we discuss distance, we are referring to variable distance, which determines the complexity and interconnectedness of the initial “atom” within it.

In this paper, we explore the prospect of generating a plethora of amazing designs using an algorithm we discovered that extends, in many ways, the Menger fractal constructions.

In a previous work, it was discovered that the Sierpinski tetrahedron with Stella Octangula has a unique property in that it can be used to fit both polyhedra made of tetrahedra, such as the Sierpinski tetrahedron (which can be replaced with a Stella Octangula), and polyhedra containing cubes, such as Menger Sponge. This is because the vertices of the Stella in a plane form a square, and the Stella occupies the volume of a cube. Thus, all figures with cubes can be substituted by Stella.

Therefore, to make a Menger Sponge with a Stella Octangula, first we needed an algorithm to build it. The first idea was to encode in a vector list the existence or not of the cube in each of the 9 positions where it could be. As the program is written in Python, the nested vector list is written as follows:

a= [[[1,1,1],[1,0,1],[1,1,1]],[[1,0,1],[0,0,0],[1,0,1]],[[1,1,1],[1,0,1],[1,1,1]]] (*Menger*)

which reflects a Menger Sponge. In each of the cubic places we place the result of the previous iteration according to whether the corresponding number of the vector list is 1 or 0. The first three levels of a Menger Sponge with Stella Octangula as the initial "atom" can be seen in Figure 1.

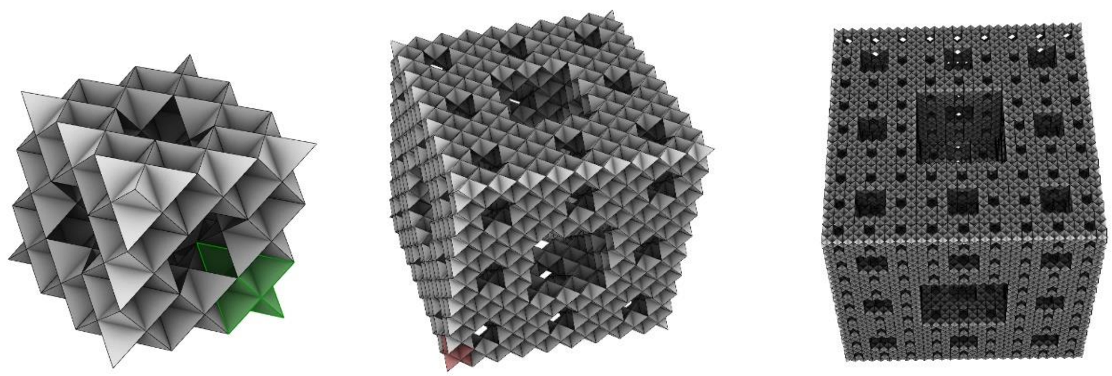

Figure 1: *From left to right, the first three levels of the Menger Sponge are represented with Stella Octangula as the initial "atom".*

The ‘dual’ Menger figure

The ‘dual’ Menger figure is obtained by replacing each of the initial “atoms” in the Menger Sponge with a space and the hole with an “atom”. By changing the 1's to 0's and the 0's to 1's in the Menger figure, we can create a "dual" Menger figure. This means that if the original Menger figure is a cube, the dual figure will be an octahedron (the dual of a cube). This holds true whether the distance between the figures is maintained in each iteration and whether the figures are allowed to intersect.
Therefore, many different matrices can be created. For example, the following one:
a=[[[0,0,0],[0,1,0],[0,0,0]],[[0,1,0],[1,1,1],[0,1,0]],[[0,0,0],[0,1,0],[0,0,0]]] (Menger 2)
To illustrate this, we apply this vector list and add an Escher’s Solid polyhedron, and the results can be seen in Figure 2.

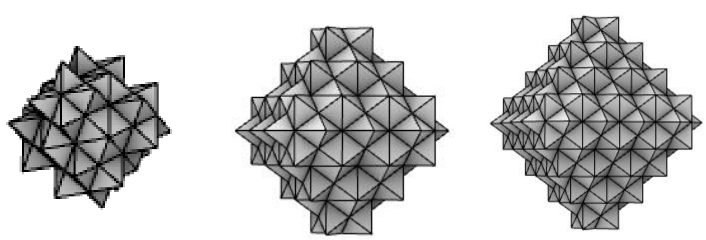

Figure 2: *Menger “dual” figure (with Escher’s Solid polyhedron) at level 1 (left), 2 (center), and 3 (right). Without change distances in the process of recursion.*

If we change the vector list again and use a truncated octahedron as initial “atom”, the result is shown in Figure 3. Interesting only with distances that not change in each level:

a= [[[0,1,0],[0,0,0],[0,1,0]],[[0,0,0],[1,0,1],[0,0,0]],[[0,1,0],[0,0,0],[0,1,0]]](*Menger 3*)

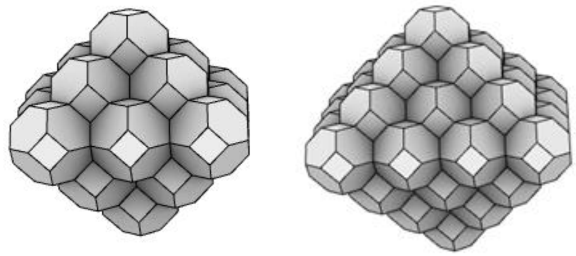

Figure 3: *Modified Menger figure (with truncated octahedrons) at levels 2 (left) and 3 (right).*

In Figure 4, we can see a "Dual" Menger figure (made up of cube "atoms") that is either constructed with a distance on which figures are contiguously touching each other in a the pure Menger style, or with a distance equal in each level, being able to intersect the figures in each new level. From now on, these two types of distance can be referred to as distance type 1 and distance type 2, respectively.

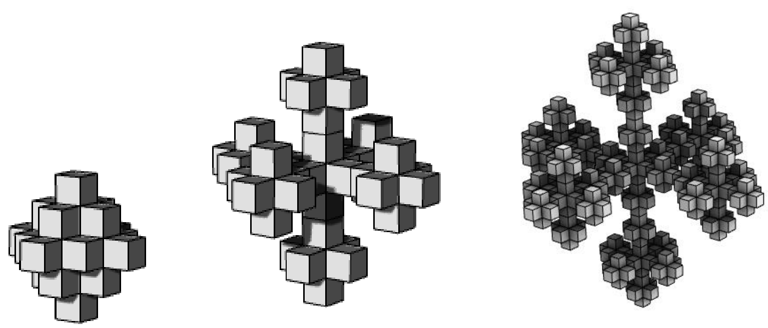

Figure 4: *Menger “dual” figure with distance type 1 (center and right) and type 2 (left) d=side.*

Playing games with distances

As we briefly mentioned above, one way to "play" with Menger-type figure configuration is to vary the distance between the “atoms” as the level increases. By doing this, it is possible to create a wide range of different Menger figures, each with its own unique properties. For example, increasing the distance between the “atoms” at higher levels can result in a Menger figure with a more open, airy structure, while decreasing the distance between the small cubes can result in a Menger figure with a more compact, dense structure. These distance changes offer figures in a non-classical Merger style. In Figure 5 we show the different results using an Escher’s solid polyhedron and different distances.

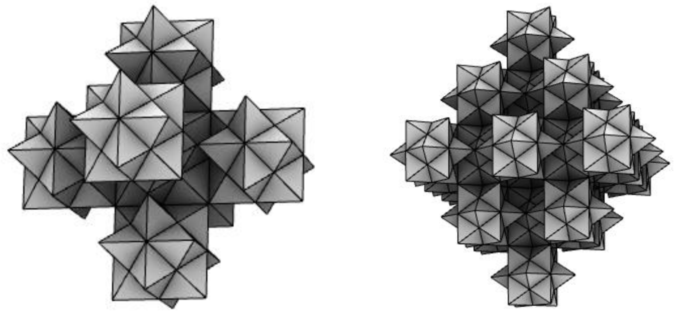

Figure 5: *Menger "dual" figure at level 1 (left) and 2 (right) and distance type 1.*

Thus, polyhedral that tessellate space (cube, Escher's solid, octahedron cube, e.g.) are relevant because they allow us to make figures that fit together at the appropriate distance, without leaving gaps. If we extend the distance or consider the fractal level to calculate the distance, as in the Menger figure itself, we will obtain new and different figures. And even by controlling the different parameters we can build infinite types of Menger Vector list fractals, for example: Allowing the base polyhedral overlap, make them touch each other only on one side or on one vertex... etc. In this way, a whole world of possibilities unfolds.

We use an algorithm that makes recursive figures increasingly larger. The result can be adjusted by simply adding a final scaling down. Figure 6 shows different variations of the Menger Sponge using cubes and changing distances. We can see if at each level the distance between "atoms" changes, in this way (i*10*3**n, j*10*3**n, k*10*3**n) depending on the three-dimensional indices (i, j, k) and the level n or in the other two images it is always n=1.

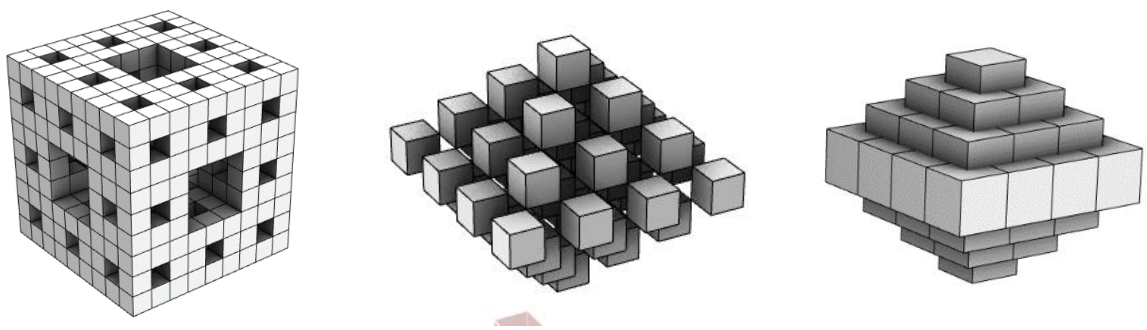

Figure 6: *Classical Menger Sponge (left) and its dual modifications changing the distances randomly.*

Based on these variations, it can be concluded that there are an enormous number of variables that can be "played" with. It is beyond the scope of this article to calculate the concrete number of possible variations following the established rules. but it is certain that we could create numerous other exploitable and aesthetically appealing figures. We should bear in mind that many unproductive rules such as the null rule (all zero) or rule 1 (all 1) should be avoided.

As a final sample, a Menger structure at level 2 is shown in Figure 7. The major peculiarity of his figure is the initial "atom", a compound of Stella Octangula, which is itself a complex figure.

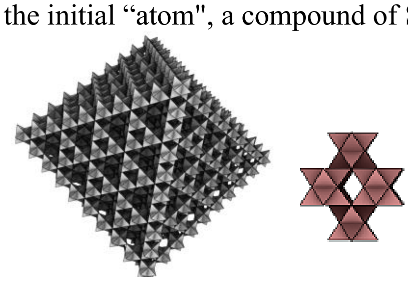

Figure 7: *Menger "dual" figure at level 2, using a compound of Stella as initial 'atom'.*

The Menger-Diaz fractals

Some tangential challenges have already been raised in relation to this issue, such as the Sierpinski Carpet in 2D, where the program and algorithms that model them are the same as in 3D, with the third coordinate set to 0 [4]. Others, such as The Jerusalem cube is something more complicated [5], but it is possible to find its algorithm. Many, many other versions (we might even say infinite) can be constructed using variants of this algorithm. As an example, in Figure 8 we can see several modified Merger Sponges created by extending the vector list.

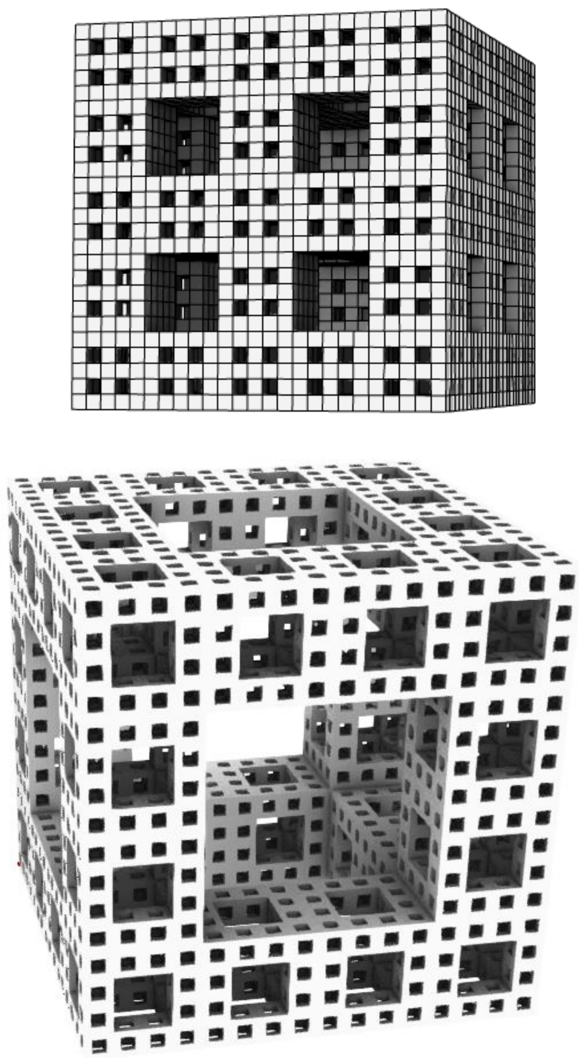

Figure 8: *Modified Menger-Diaz Sponges at level 2.*

However, recently, an interesting variation called Mosely snowflake was developed [6]. The Mosely snowflake is a cube-based fractal with corners recursively removed; therefore, we can then draw the following vector list.

a= [[[0,1,0],[1,1,1],[0,1,0]],[[1,1,1],[1,0,1],[1,1,1]],[[0,1,0],[1,1,1],[0,1,0]]](Mosely)

Consequently, we found that the algorithm covers some other examples of variations, as shown in Figure 9.

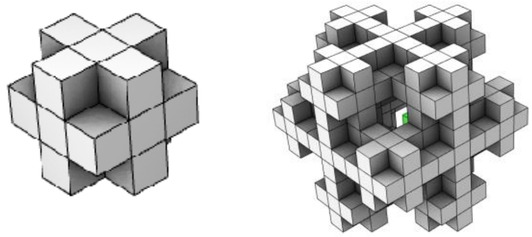

Figure 9: *Mosely figures at level 1 (left) and 2 (right).*

Following some of the guidelines outlined throughout the article, we can create a wide range of variants, which from now on we will refer to as Menger-Diaz fractals. The first, the Menger "dual". In others, the algorithm adopted is as follows:

a=[[[1,0,1],[0,1,0],[1,0,1]],[[0,1,0],[1,0,1],[0,1,0]],[[1,0,1],[0,1,0],[1,0,1]]] (*Menger-Diaz 2*)

The number of figures that can be developed from this rule is manifold. The first examples of Menger-Diaz figures are shown in Figure 10.

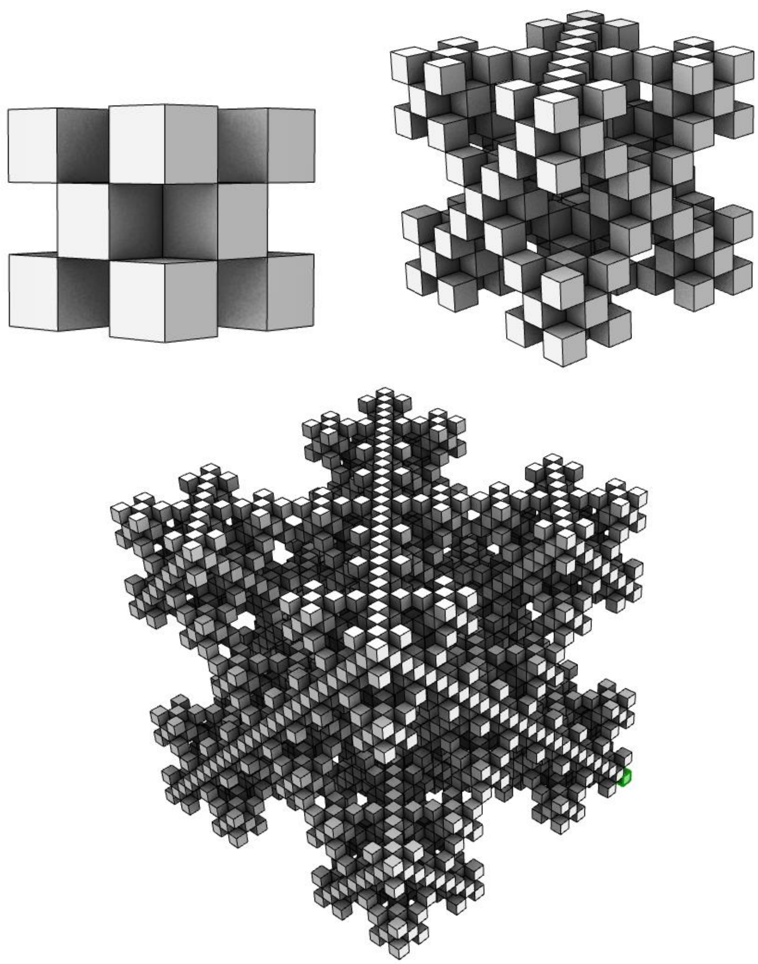

Figure 10: *Menger-Diaz figures at level 1 (left), level 2 (right), and level 3 (bottom).*

If we modify the vector lists, we can still create works of art in a simple and mechanical way (examples below).

a=[[[1,0,1],[0,0,0],[1,0,1]],[[0,0,0],[0,1,0],[0,0,0]],[[1,0,1],[0,0,0],[1,0,1]]] (*Menger-Diaz 3*)
a=[[[0,1,0],[1,0,1],[0,1,0]],[[1,0,1],[0,1,0],[1,0,1]],[[0,1,0],[1,0,1],[0,1,0]]] (*Menger-Diaz 4*)

More possible variants, such as the one shown in Figures 11 (rule 3) and 12 (rule 4), are shown below. In Figure 11 a very familiar fractal appears when we oversample the figure into a horizontal plane.

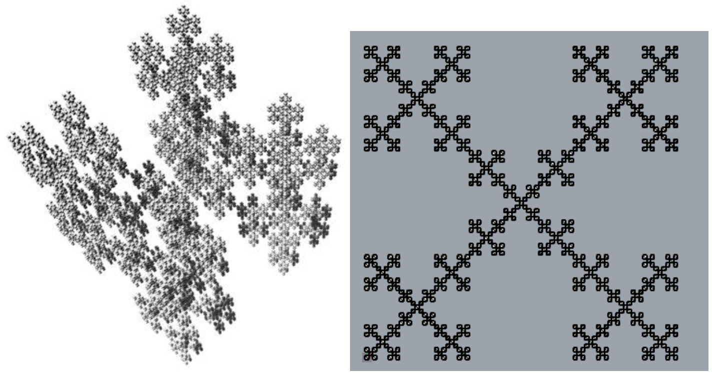

Figure 11: *Menger-Diaz figures at level 4 and its horizontal plane (right).*

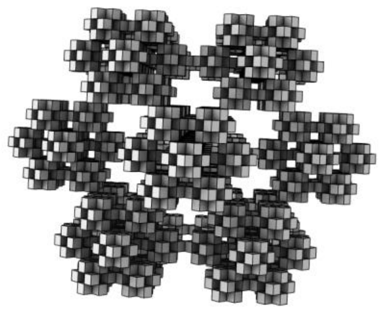

Figure 12: *Menger-Diaz figure at level 3.*

Summary and Conclusions

In general, playing with the initial 'atom', levels, and distances in Menger-type figure configuration can be a powerful tool for exploring the properties and behavior of these fascinating geometric figures. By experimenting with different distance configurations, it is possible to gain a deeper understanding of Menger-type figures and its many interesting properties.
Based on this method, Menger-Diaz variations have been established. Menger-Diaz variations are a new variation of Menger-type fractals that is created by modifying the way in which the initial cubes are removed at each recursion level. This results in a shape with unique properties and characteristics.

In the field of art, Menger-like fractal structures can be used to create visually striking and complex works of art. These structures have an infinite number of faces and an infinitely complex surface, allowing for great deal of creativity and flexibility in the design process. Artists can use these structures as the basis for paintings, sculptures, digital art, and other media and can explore a wide range of themes and styles using these structures as a starting point.

In other fields, Menger-like fractal structures may also have potential applications in areas such as architecture, engineering, and computer science. For example, these structures could be used to design more efficient and sustainable buildings or to create new algorithms and data structures. Overall, the development of new Menger-like fractal structures has the potential to significantly impact and enrich a wide range of fields and disciplines.

Other consequences are in the field of classification of fractals: for example, the Sierpinski Tetrahedron is also a Menger-Diaz fractal. In a new article we shall explain this and another themes.

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