Artificial Neural Networks: Integral Equations: Physics Informed Neural Networks

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ABSTRACT

In this research paper, a novel model of artificial neuron (in the spirit of spiking neuron) is summarized. It is discussed, how integral equations naturally arise in associative memory based on such a model of artificial neuron. This research paper mainly proposes, the utilization of Physics Informed Neural Networks (PINNs) for solving integral equations. We expect to utilize PINNs for data driven discovery of integral equations. It is expected that this research paper will lead to new research on solving non-linear integro-differential equations

1. **Introduction:**

In an effort to model the biological neural network, researchers innovated the research area of Artificial Neural Networks (ANNs). Alongwith McCulloch-Pitts model of neuron, various other models such as the "Spiking Neuron" model are proposed. Various types of such neurons are interconnected resulting in feedforward neural networks, recurrent neural networks etc. In the beginning, the feedforward neural networks had smaller number of hidden layers. Researchers realized that novel types on ANNs, called, "Convolutional Neural Networks (CNNs) with large number of hidden layers were providing better than human accuracy in classification and other applications. Without knowledge of the spiking neuron model, the author proposed a novel model of neuron in continuous time as well as discrete time. The main idea was to consider the neuron as a distributed element (rather than lumped element) modeled as a linear filter with the associated impulse response in discrete/continuous time. The related research was published in [3], [5].

In recent years, researchers proposed artificial neural networks to determine/approximate the solutions of partial differential equations. Such networks are called Physics Informed Neural Networks (PINNs). They are found to be quite successful in applications such as Computational Fluid Dynamics (CFD). The other conceived the idea of solving LINEAR/NONLINEAR INTEGRAL EQUATIONS using ANNs in the spirit of PINNs. Specifically, nonlinear integral equations based on Fredholm and Volterra integral equations naturally arise in the models of ANNs, namely associative memories. In general, researchers are actively interested in solving nonlinear integral equations arising in various other applications. More generally, we expect PINN based solutions of linear/nonlinear integral equations to find interesting applications

This research paper is organized as follows. In Section 2, a novel model of artificial neuron innovated by the author is described. In Section 3, a novel model of associative memory is proposed. It is shown how nonlinear integral equations naturally arise in such an Artificial Neural Network. In Section 4, solution of linear as well as nonlinear integral equations using Physics Informed Neural Networks is briefly discussed. The research paper concludes in Section 5.

2. Novel Model of Neuron: Convolution and Correlation Operations:

Consider the case, where the input signal is defined over finite time. Naturally, the output of associated artificial neuron is defined by a function . We consider both the continuous time and discrete time signals. Also, the synapse of an artificial neuron is considered as a distributed element rather than a lumped element with a single weight value [3]. Specifically, the synapse is modeled as a linear filter with the associated impulse response. As in the case of traditional neurons, the activation function is considered to be the signum function.

We now describe, a model of artificial neuron in continuous time. Let

$$u_j(t) \ldots\ldots jth\ input\ function\ to\ the\ artifical\ neuron$$
$$h_j(t) \ldots\ldots impulse\ response\ of\ j\ the\ synapse$$
$$T(t) \ldots. Threshold\ function\ at\ the\ artificial\ neuron$$
$$y(t) \ldots\ldots Output\ of\ the\ artificial\ neuron$$
$$M \ldots number\ of\ input\ functions\ to\ the\ artificial\ neuron.$$

The output of the artificial neuron is computed in the following manner:

$$y(t) = Signum \left\{ \sum_{j=1}^{M} h_j(t) \ * \ u_j(t) \ - \ T(t) \right\}, \ldots\ldots\ldots (1)$$

$$where\ *\ denotes\ the\ CONVOLUTION\ operation. In\ this\ case\ the\ functions\ involved\ are\ the\ impulse\ response\ of\ synapse\ and\ the\ input\ to\ neuron.$$

$Note$: Networks of such artificial neurons were utilized in Spiking neuron based ANNs. Several interesting applications were found for such ANNs.

$NOTE$: We can consider the case where the artificial neuron operates on the functions defined over discrete time units. In such case *'t'* is a discrete time variable. Thus, when *'t'* is a discrete time variable, the synapse could be modeled by a *finite impulse response linear filter.* Specifically, the output of discrete time artificial neuron is given by

$$y[n] = Signum \left\{ \sum_{j=1}^{M} h_j[n] \ * \ u_j[n] \ - \ T[n] \right\}, where\ \ldots\ldots\ldots (2)$$

$$h_j[n]\ could\ correspond\ to\ a\ finite\ impulse\ response\ filter.$$

3. Novel Associative Memories: Integral Equations:

The following research problem was formulated by Wyner [6]. The open research problem was formulated with application to “Signal Design for Magnetic and Optical recording Channels”:

"Given a linear time invariant system in continuous time, consider the class of input signals to such a system defined over a finite time interval [0, T]. Let the input signals be also bounded in amplitude L i.e. | u(t) | $\leq L$. Determine the sub-class of such input signals which maximize the output energy i.e. $\int_0^T y^2(t)dt$.

It was proved by the author and independently by Honig that the necessary condition on the input signal is given by

$$x(t) = \ Signum \ \left(\int_0^T R(t-u)\, x(u)\, du \right), \qquad \text{..........(3)}$$

where $R(.)$ *is the autocorrelation function of the impulse response signal. The condition thus corresponds to a signed Fredholm integral equation.* It is clearly a non-linear integral equation. Using successive approximation scheme, it is proposed to approximate the optimal input signal.

It is well known that Mc-Culloch-Pitts neuron motivated the Hopfield Neural Network.

Such a neural network acts as an associative memory, the so called Hopfield Associative Memory (HAM). The associated convergence Theorem shows that the ANN acts as a local optimization device based on a quadratic energy function. The continuous time version of Hopfield Associative Memory was also well studied. In [4], the author proposed a novel associative memory where the neuronal model is based on equation (3). A network of such artificial neurons are connected with the synapases modeled as linear time invariant filters. We discretize such a system operating in continuous time. We proposed the convergence Theorem associated with such an associative memory.

In the parallel mode of operation of such an ANN, the state i.e. $\bar{X}$ (t) vector satisfies the following NONLINEAR SIGNED INTEGRAL EQUATION when the ANN converges to a STABLE STATE

$$\bar{X}(t) \ = \ SIGN \ \left(\int_0^T \bar{R}\,(t-u)\, \bar{X}(u)\, du - \ \bar{T}(t) \right).$$

The above equation constitutes a VECTOR-MATRIX NON-LINEAR SIGNED FREDHOLM INTEGRAL EQUATION. In general, linear as well as nonlinear integral equations naturally arise in the design and analysis of Artificial Neural Networks.

More importantly, solving linear and non-linear integral equations is an important problem research problem in Science, Engineering and other disciplines.

4. **Physics Informed Neural Networks: Solution of Integral Equations:**

Physics informed neural networks are ANNs trained to solve supervised learning tasks while respecting the associated dynamics described by linear as well as nonlinear integral equations for which closed form solutions are not possible [2]. Such an ANNs are utilized to solve two classes of problems: Data Driven Solutions and Data Driven discovery of partial differential equations. It is well known that deep feed forward neural networks are capable of universal function approximation. This well known fact is utilized in training Physics Informed Neural Networks (PINNs) for solving non-linear partial differential equations without resorting to standard assumptions such as linearization. PINNs utilize latest advances in automatic differentiation [1]. The loss function of the ANN is based on the nonlinear differential equation being approximated.

- **INNOVATIVE IDEA**:
  The novel idea is to approximate integral of a function using approximated using the well known schemes for numerical integration such as Simpson's rule (and recent advances in automatic integration). The loss function is once again based on the integral equation (possibly nonlinear) being approximated. We expect intense research activity based on the above idea for utilization of PINNs for solving (data driven methods) even NON-LINEAR INTEGRAL EQUATIONS and data driven discovery of integral equations

5. CONCLUSIONS:

In this research paper, the innovative idea of utilization of Physics Informed Neural Networks (PINNs) for solving non-linear integral equations. We expect detailed efforts to be progressed by the Artificial Neural Networks community

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