Complexity of Admissible Rules in the Implication-Negation Fragment of Intuitionistic Logic

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Abstract

The goal of this paper is to study the complexity of the set of admissible rules of the implication-negation fragment of intuitionistic logic IPC. Surprisingly perhaps, although this set strictly contains the set of derivable rules (the fragment is not structurally complete), it is also PSPACE-complete. This differs from the situation in the full logic IPC where the admissible rules form a co-NEXP-complete set.

1 Introduction

Following Lorenzen [12], a rule is said to be *admissible* for a logic (understood as a finitary structural consequence relation) if it can be added to a proof system for the logic without producing any new theorems. While the admissible rules of classical propositional logic CPC are also *derivable* – that is, CPC is *structurally complete* – this is not the case for non-classical (modal, many-valued, substructural, intermediate) logics in general (see, e.g., [16, 14, 2]). In particular, the study of admissible rules was stimulated by the discovery of admissible but underivable rules of intuitionistic propositional logic IPC such as the independence of premises rule:

$$\neg p \to (q \vee r) \;/\; (\neg p \to q) \vee (\neg p \to r).$$

The decidability of the set of admissible rules of IPC, posed as an open problem by Friedman in [4], was answered positively by Rybakov, who demonstrated also that this set has no finite *basis* (understood as a set of admissible rules that added to IPC produces all admissible rules) [16]. Nevertheless, following a conjecture by de Jongh and Visser, Iemhoff [7] and Rozière [15] established independently that an infinite basis is formed by the family of "Visser rules" ($n = 2, 3, \ldots$):

$$(\bigwedge_{i=1}^{n}(q_i \to p_i) \to (q_{n+1} \vee q_{n+2})) \vee r \;/\; \bigvee_{j=1}^{n+2}(\bigwedge_{i=1}^{n}(q_i \to p_i) \to q_j) \vee r.$$

More generally, the work of Rybakov [16] and Ghilardi [5, 6] has led to a reasonably comprehensive understanding of structural completeness and admissible rules for broad classes of intermediate and modal logics. Kripke-frame based characterizations of *hereditarily structurally*

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complete (i.e., each extension of the logic is structurally complete) intermediate logics and transitive modal logics have been obtained by Citkin and Rybakov [3, 16]. Bases have been provided for certain intermediate logics by Iemhoff [8] and for transitive modal logics by Jeřábek [11], and Gentzen-style proof systems have been developed for these logics by Iemhoff and Metcalfe [10, 9]. Note, moreover, that in these cases, admissibility is characterized in the wider setting of multiple-conclusion rules, where, as the name suggests, many conclusions as well as many premises are permitted. For example, a paradigmatic example of a multiple-conclusion rule admissible in intuitionistic logic but not classical logic is the disjunction property, which may be formulated as:

$$p \vee q \ / \ p, q.$$

Mints demonstrated structural completeness for implication-less fragments of IPC and showed moreover that any admissible underivable rule of IPC must contain both implication and disjunction [13]. Curiously, however, as observed by Wroński [18], the *implication-negation fragment* (equivalently, the implication-falsity fragment) – the logic of bounded BCKW-algebras – is not structurally complete. Consider, e.g., the following rule:

$$((\neg\neg p \to p) \to r), ((\neg\neg q \to q) \to r), (p \to \neg q) \ / \ r.$$

This rule is not derivable in IPC and therefore not in any of its fragments. The rule is also not admissible in IPC. However, it *is* admissible in the implication-negation fragment of this logic.

The goal of this short paper is to study the complexity of the set of admissible rules of the implication-negation fragments of IPC. To achieve this goal we first describe a basis for the set of admissible rules. The results and techniques used in this work can be extended to all axiomatic extensions of the fragment and are studied in detail in the submitted paper [1], where the reader may also find all omitted proofs.

2 Preliminaries

Let us denote by L the implication-negation fragment of intuitionistic logic IPC. The basic connectives of L are taken to be $\to$ and $\bot$, defining $\neg\varphi =_{\rm def} \varphi \to \bot$ and $\top =_{\rm def} \bot \to \bot$. We abbreviate $\varphi_1 \to (\varphi_2 \to (\ldots \to (\varphi_n \to \psi)\ldots)$ by $\varphi_1 \to \varphi_2 \to \ldots \to \varphi_n \to \psi$ or $\vec{\varphi} \to \psi$. For $\vec{\varphi} = \emptyset$, we understand $\vec{\varphi} \to \psi$ to be the formula ψ. We use Γ, Π, Δ without further comment to denote finite sets of formulas and p, q, r to denote propositional variables. Since by the Glivenko theorem, a set of formulas is L-consistent if and only if $\Gamma \not\vdash_{\rm CPC} \bot$, we drop the prefix and speak just of consistency.

A *rule* for L is an ordered pair (Γ, Δ), written $\Gamma \ / \ \Delta$, where $\Gamma \cup \Delta$ is a finite set of formulas, called *single-conclusion* if $|\Delta| = 1$ and *multiple-conclusion* in general. We write '$\Gamma \ / \ \varphi$', 'Γ, Δ', and 'Γ, φ' for, respectively, '$\Gamma \ / \ \{\varphi\}$', '$\Gamma \cup \Delta$', and '$\Gamma \cup \{\varphi\}$'.

A logic such as L is traditionally understood as a set of single-conclusion rules (as our logic is finitary we can ignore rules with infinite sets of premises), and we write $\Gamma \vdash_{\rm L} \varphi$ instead of $\Gamma \ / \ \varphi \in {\rm L}$. As we are interested in multiple-conclusion rules we need to introduce a notion of multi-conclusion logic: an *m-logic* is a set M of rules (writing $\Gamma \vdash_{\rm M} \Delta$ instead of $(\Gamma, \Delta) \in {\rm M}$) satisfying for all finite sets of formulas $\Gamma, \Gamma', \Delta, \Delta'$ and formula φ:

1. $\varphi \vdash_{\rm M} \varphi$
2. if $\Gamma \vdash_{\rm M} \Delta$, then $\Gamma, \Gamma' \vdash_{\rm M} \Delta', \Delta$
3. if $\Gamma, \varphi \vdash_{\rm M} \Delta$, and $\Gamma' \vdash_{\rm M} \varphi, \Delta'$ then $\Gamma, \Gamma' \vdash_{\rm M} \Delta', \Delta$

4. if $\Gamma \vdash_{\mathrm{M}} \Delta$, then $\sigma(\Gamma) \vdash_{\mathrm{M}} \sigma(\Delta)$ for each substitution σ.

We introduce a particular multiple-conclusion variant of L as:

$$\mathrm{L_m} = \{\Gamma \;/\; \Delta \mid (\exists \varphi \in \Delta)(\Gamma \vdash_{\mathrm{L}} \varphi)\}.$$

Finally, we formally introduce the crucial concepts of the paper: a rule $\Gamma \;/\; \Delta$ is said to be *derivable* if $\Gamma \vdash_{\mathrm{L_m}} \Delta$, and *admissible*, written $\Gamma \mathrel{|\!\!\sim}_{\mathrm{L}} \Delta$, if for each substitution σ: whenever $\vdash_{\mathrm{L}} \sigma\varphi$ for all $\varphi \in \Gamma$, also $\vdash_{\mathrm{L}} \sigma\psi$ for some $\psi \in \Delta$. Clearly $\mathrel{|\!\!\sim}_{\mathrm{L}}$ is an m-logic and contains $\mathrm{L_m}$. A *basis* for $\mathrel{|\!\!\sim}_{\mathrm{L}}$ over $\mathrm{L_m}$ is a set of rules S such that $\mathrel{|\!\!\sim}_{\mathrm{L}}$ is the smallest m-logic which contains both $\mathrm{L_m}$ and S.

3 Basis of Admissible Rules

A basis for the admissible rules of L will consist of the following "Wroński rules" ($n \in \mathbb{N}$):

$$(W_n) \quad (\vec{p} \to \bot) \;/\; (\neg\neg p_1 \to p_1), \ldots, (\neg\neg p_n \to p_n)$$

Note that in the case of $n = 0$ (useful for technical reasons) (W_0) is $\bot \;/\; \emptyset$ and is admissible but not derivable.

Lemma 3.1 ([1])**.** *(W_n) is admissible for each $n \in \mathbb{N}$.*

Observe on the other hand that these rules may not be admissible in fragments of an intermediate logic containing $\wedge$ or $\vee$ as well as $\to$ and $\bot$. In particular, for IPC, let $\sigma p_1 = p \wedge \neg q$ and $\sigma p_2 = q$. Then $\vdash_{\mathrm{IPC}} \sigma(p_1 \to p_2 \to \bot)$ but $\nvdash_{\mathrm{IPC}} \sigma(\neg\neg p_1 \to p_1)$ and $\nvdash_{\mathrm{IPC}} \sigma(\neg\neg p_2 \to p_2)$.

Let us denote by $\mathrm{L_m^W}$ the least m-logic containing $\mathrm{L_m}$ and the rules (W_n) for each $n \in \mathbb{N}$. We show that these rules form a basis for $\mathrel{|\!\!\sim}_{\mathrm{L}}$ over $\mathrm{L_m}$, i.e., $\mathrm{L_m^W} = \mathrel{|\!\!\sim}_{\mathrm{L}}$. (Notice that the inclusion $\mathrm{L_m^W} \subseteq \mathrel{|\!\!\sim}_{\mathrm{L}}$ follows immediately from the previous lemma).

The first step of our strategy will be to 'reduce' the question of the admissibility of any rule to the admissibility of rules of a certain basic form. Let us call a formula having one of the following forms *simple*:

(i) $\vec{p} \to \bot$

(ii) $\vec{\psi} \to r$ where each member of $\vec{\psi}$ is of the form $p \to q$ or p.

The next lemma formalizes the 'reduction' idea. Its proof is based on replacing non-simple formulas by formulas which are, in a sense, 'more simple'. We provide a suitable complexity measure and show that our 'simplification' procedure decreases it, thus obtaining that the process terminates.

Lemma 3.2 ([1])**.** *For any rule $\Gamma \;/\; \Delta$, there exists a finite set of simple formulas Π such that:*

1. $\Gamma \vdash_{\mathrm{L_m^W}} \Delta$ *iff* $\Pi \vdash_{\mathrm{L_m^W}} \Delta$
2. $\Gamma \mathrel{|\!\!\sim}_{\mathrm{L}} \Delta$ *iff* $\Pi \mathrel{|\!\!\sim}_{\mathrm{L}} \Delta$.

Next, our strategy will be to to construct a set Ψ_Γ of sets of formulas containing Γ and reduce admissibility $\Gamma \mathrel{|\!\!\sim}_{\mathrm{L}} \Delta$ to derivability of $\Gamma' \vdash_{\mathrm{L_m}} \Delta$ for all $\Gamma' \in \Psi_\Gamma$, which in turn we further reduce to derivability of $\Gamma \vdash_{\mathrm{L_m^W}} \Delta$. Roughly speaking the Ψ_Γ will contain all possible exhaustive applications of the rules from W to Γ, i.e, we consider all sets of variables $\vec{p}$ such that

$$\Gamma \vdash_{\mathrm{L}} \vec{p} \to \bot$$

and obtain new sets of formulas by adding for each such $\vec{p}$ a formula $\neg\neg p \to p$ for some $p \in \vec{p}$.

Let us elaborate these ideas in detail. By $\mathrm{Var}(\Gamma)$ we denote the set of variables occurring in Γ. We enumerate the sets of variables $X \subseteq \mathrm{Var}(\Gamma)$ such that $\Gamma \cup X \vdash_{\mathrm{L}} \bot$ as $X_1, \ldots, X_n$, and define the following sequence:

1. $\Psi_0 = \{\emptyset\}$
2. $\Psi_i = \{\Pi \cup \{\neg\neg p \to p\} \mid \Pi \in \Psi_{i-1} \text{ and } p \in X_i\}$ for $i = 1 \ldots n$.

Let $\Psi_\Gamma = \{\Pi \cup \Gamma \mid \Pi \in \Psi_n\}$. Note that if Γ is inconsistent, then $X_i = \emptyset$ for some $i \in \{1, \ldots, n\}$ and it follows that $\Psi_j = \emptyset$ for $i \leq j \leq n$, and so $\Psi_\Gamma = \emptyset$.

The two reductions mentioned above are formalized as:

Lemma 3.3 ([1])**.** *Let Γ be a finite set of simple formulas. If $\Gamma \mathrel{|\!\sim}_{\mathrm{L}} \Delta$, then $\Gamma' \vdash_{\mathrm{L_m}} \Delta$ for all $\Gamma' \in \Psi_\Gamma$.*

Lemma 3.4 ([1])**.** *Let Γ be a finite set of simple formulas. If $\Gamma' \vdash_{\mathrm{L_m}} \Delta$ for all $\Gamma' \in \Psi_\Gamma$, then $\Gamma \vdash_{\mathrm{L}_\mathrm{m}^W} \Delta$.*

Theorem 3.5. *W is a basis for $\mathrel{|\!\sim}_{\mathrm{L}}$ over $\mathrm{L_m}$.*

Proof. We have to show that $\Gamma \mathrel{|\!\sim}_{\mathrm{L}} \Delta$ iff $\Gamma \vdash_{\mathrm{L}_\mathrm{m}^W} \Delta$. One direction was established in Lemma 3.1. To prove the second one suppose that $\Gamma \mathrel{|\!\sim}_{\mathrm{L}} \Delta$. By Lemma 3.2, we can construct a finite set of simple formulas Π such that $\Pi \mathrel{|\!\sim}_{\mathrm{L}} \Delta$. But then by Lemma 3.3, $\Pi' \vdash_{\mathrm{L_m}} \Delta$ for all $\Pi' \in \Psi_\Pi$. Hence by Lemma 3.4, $\Pi \vdash_{\mathrm{L}_\mathrm{m}^W} \Delta$ and Lemma 3.2 completes the proof that $\Gamma \vdash_{\mathrm{L}_\mathrm{m}^W} \Delta$ □

4 Complexity

We start by defining set $F(\Gamma)$ of sets of atoms and set Ψ'_Γ of sets of formulas:

$$\begin{aligned} F(\Gamma) &= \{Y \subseteq \mathrm{Var}(\Gamma) \mid (\forall X)(\Gamma, X \vdash_{\mathrm{L}} \bot \Rightarrow X \cap Y \neq \emptyset)\} \\ \Psi'_\Gamma &= \{\Gamma \cup \{\neg\neg p \to p \mid p \in Y\} \mid Y \in F(\Gamma)\}. \end{aligned}$$

Lemma 4.1. *Let Γ be a set of simple formulas. Then $\Gamma \mathrel{|\!\sim}_{\mathrm{L}} \Delta$ iff $\Pi' \vdash_{\mathrm{L_m}} \Delta$ for each $\Pi' \in \Psi'_\Gamma$.*

Proof. Suppose first that $\Gamma \ / \ \Delta$ is admissible. Notice that for each $\Pi' \in \Psi'_\Gamma$ there is $\Pi \in \Psi_\Gamma$ such that $\Pi' \supseteq \Pi$, thus the left-to-right direction follows by Lemma 3.3. Since $\Psi_\Gamma \subseteq \Psi'_\Gamma$, the reverse direction follows by Lemmas 3.4 and 3.1. □

Lemma 4.2. *Let Γ be a set of simple formulas. Then deciding $\Pi \in \Psi'_\Gamma$ is solvable in non-deterministic polynomial time (with respect to the size of Γ).*

Proof. We show that $\Pi = \Gamma \cup \{\neg\neg p \to p \mid p \in Y\} \in \Psi'_\Gamma$ iff $\Gamma, \mathrm{Var}(\Gamma) \setminus Y \nvdash_{\mathrm{L}} \bot$, which reduces the problem to satisfiability in classical logic. From the construction of Ψ'_n we know that $\Pi \in \Psi'_\Gamma$ iff $Y \in F(\Gamma)$. To complete the proof we show that $Y \in F(\Gamma)$ iff $\Gamma, \mathrm{Var}(\Gamma) \setminus Y \nvdash_{\mathrm{L}} \bot$ For the first direction, assume that $Y \in F(\Gamma)$. Since $(\mathrm{Var}(\Gamma) \setminus Y) \cap Y = \emptyset$, we obtain $\Gamma, \mathrm{Var}(\Gamma) \setminus Y \nvdash_{\mathrm{L}} \bot$. For the converse direction, assume that $\Gamma, \mathrm{Var}(\Gamma) \setminus Y \nvdash_{\mathrm{L}} \bot$ and $\Gamma, X \vdash_{\mathrm{L}} \bot$. Then $X \not\subseteq \mathrm{Var}(\Gamma) \setminus Y$. I.e., $X \cap Y \neq \emptyset$. □

Theorem 4.3. *The set of admissible rules of the implication-negation fragment of intuitionistic logic is* PSPACE-*complete.*

Proof. PSPACE-hardness follows from the fact that the set of theorems for this fragment of IPC is already PSPACE-hard [17]. Next, observe that Lemma 3.2 reduces our problem to the problem of admissibility of rules with simple set of premises. By inspection of the proof of this lemma in [1] we see that this reduction is polynomial. To solve this problem we use the contrapositive version of Lemma 4.1. Consider a rule $\Gamma \ / \ \Delta$ with simple premises. First we observe that all $\Pi \in \Psi'_\Gamma$ are of polynomial size with respect to Γ. Thus, to show that $\Gamma \ / \ \Delta$ is *not* admissible we can nondeterministically guess some $X \subseteq \mathrm{Var}(\Gamma)$ and $\varphi \in \Delta$ and check whether $\Pi = \Gamma, \{\neg\neg p \to p \mid p \in X\} \in \Psi'_\Gamma$ and the IPC-nonderivability of $\Pi \ / \ \varphi$, a problem in PSPACE. Finally, we use the fact that coNPSPACE = PSPACE. $\square$

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