A general framework for geometric dualities for varieties of algebras

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Abstract

We set up a framework that subsumes many important dualities in mathematics (Birkhoff, Stone, Priestly, Baker-Beynon, etc.) as well as the classical correspondence between polynomial ideals and affine varieties in algebraic geometry. Our main theorems provide a generalisation of Hilbert's *Nullstellensatz* to any (possibly infinitary) variety of algebras. The common core of the above dualities becomes then clearly visible and sets the basis to a canonical method to seek for a geometric dual to any given variety of algebras.

1 Introduction.

The starting point is the observation that the Galois connection between set of terms in a fixed language, seen as polynomial evaluated on some algebra, and *vanishing* sets of these terms, can always be made functorial yielding a dual adjunction. Modulo the Axiom of Choice, one can extend this pair of functors to an adjunction between a variety and subspaces of a fixed algebra in that variety carrying a non-trivial topology induced by the Galois correspondence (aka Zariski topology). Notably, several known dualities such as Stone's dualities for Boolean algebras and distributive lattices, Priestly's duality, Stone-Gelfand's duality for real C*-algebras, can be nestled in this framework.

Similar approaches can be found in the work of E. Daniyarova, A. Myasnikov, V. Remeslennikov, where the emphasis is put on the methods from algebraic geometry which become available and in the work of Y. Diers which investigates this adjunction at a more abstract categorical level.

2 Notation.

Let V be a fixed but arbitrary (possibly infinitary) variety of algebras. We identify V with the category V of all algebras in the variety, with their homomorphisms. Throughout, μ and ν invariably denote cardinal numbers, whereas α and β invariably denote ordinal numbers. Let also $\mathscr{F}(\mu)$ the free algebra over μ generators in the variety V. Although elements of $\mathscr{F}(\mu)$ are equivalence classes of terms in the language of the variety V, we often use single terms as representatives for their equivalence classes.

If s is a term, the notation $s\big((X_\alpha)_{\alpha<\mu}\big)$ means that the (finitely many) variables occurring in s are among those in the tuple $(X_\alpha)_{\alpha<\mu}$. If $s\big((X_\alpha)_{\alpha<\mu}\big) \in \mathscr{F}(\mu)$ and $\{t_\alpha\}_{\alpha<\mu} \subseteq \mathscr{F}(\nu)$, we denote by $s\big(\ [X_\alpha \backslash t_\alpha]_{\alpha<\mu}\ \big)$ the term obtained from s by uniformly replacing each variable X_α with the term t_α. Obviously, $s\big(\ [X_\alpha \backslash t_\alpha]_{\alpha<\mu}\ \big) \in \mathscr{F}(\nu)$.

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If $A \in \mathsf{V}$ we write A^μ for the Cartesian product of μ copies of A. If $p \in A^\mu$, then $s(p)$ denotes the evaluation of the term s in the V-algebra A under the assignment $X_\alpha \mapsto \pi_\alpha(p)$, where $\pi_\alpha \colon A^\mu \to A$ is the projection onto the α^{th} coordinate, for each ordinal $\alpha < \mu$.

3 The general adjunction.

As above, let V be a fixed variety and fix any $A \in \mathsf{V}$. We define two operators $\mathbb{V}$ and $\mathbb{I}$ as functions between partially ordered sets, namely, the powersets of $\mathscr{F}(\mu) \times \mathscr{F}(\mu)$ and A^μ. The notation $\mathbb{V}_A$ and $\mathbb{I}_A$ would be more precise as the operators obviously depend on the selected algebra A. However, as A is fixed we prefer to lighten the notation by dropping the A in the names of the operators.

Definition 3.1 (The operator $\mathbb{I}$.)**.** Given $S \subseteq A^\mu$, let us define a relation $\mathbb{I}(S)$ on $\mathscr{F}(\mu)$ by stipulating that, for arbitrary terms $s, t \in \mathscr{F}(\mu)$,

$$\mathbb{I}(S) = \{(s,t) \in \mathscr{F}(\mu) \times \mathscr{F}(\mu) \mid A \models s(p) \approx t(p), \, \forall p \in S\}$$

for every $p \in S \subseteq A^\mu$. We call $\mathbb{I}(S)$ the *vanishing congruence* of S.

Definition 3.2 (The operator $\mathbb{V}$.)**.** Given $R = \{(s_i, t_i) \mid i \in I\} \subseteq \mathscr{F}(\mu) \times \mathscr{F}(\mu)$, for I an index set, the *vanishing locus* of R is

$$\mathbb{V}(R) = \{p \in A^\mu \mid A \models s_i(p) \approx t_i(p), \, \forall i \in I\}.$$

Within this general framework, no matter on the choice of A and V one always has the following:

Lemma 3.3 (Basic Galois connection)**.** *For each $S \subseteq A^\mu$ and $R \subseteq \mathscr{F}(\mu) \times \mathscr{F}(\mu)$,*

$$R \subseteq \mathbb{I}(S) \text{ if, and only if, } S \subseteq \mathbb{V}(R) .$$

In words, the functions $\mathbb{V}$ and $\mathbb{I}$ form a Galois connection.

Galois connections are basic instances of categorical adjunctions, so one may ask under which circumstance the above correspondence can be made functorial. The answer is: always! Let V_p be the category of *presented* V-algebras, i.e. algebras of the form $\mathscr{F}(\mu)/\theta$ with μ ranging among all cardinals and θ among all congruences[1].

Definition 3.4 (Definable maps)**.** Given $S \subseteq A^\mu$ and $T \subseteq A^\nu$, a function $\lambda \colon S \to T$ is *definable* if there exists a ν-tuple of terms $(l_\beta)_{\beta<\nu}$, with $l_\beta \in \mathscr{F}(\mu)$, such that

$$\lambda(\,(p_\alpha)_{\alpha<\mu}\,) = (\,l_\beta(\,(p_\alpha)_{\alpha<\mu}\,)\,)_{\beta<\nu}$$

Let us call $\mathsf{G}_{\mathsf{def}}$ the category of subsets of A^μ, with μ ranging among all cardinals, and definable maps among them. We extend the above Galois connection to a pair of functors

$$\mathscr{I} \colon \mathsf{G}_{\mathsf{def}}^{\text{op}} \to \mathsf{V}_\mathsf{p} \text{ and } \mathscr{V} \colon \mathsf{V}_\mathsf{p} \to \mathsf{G}_{\mathsf{def}}^{\text{op}} .$$

[1] If one assumes AC then V and V_p are obviously equivalent, however, as the presentation of an algebra plays a main role in our construction, we prefer to explicitly work with presented algebras as above.

Definition 3.5 (The functor $\mathscr{I}$ on objects.)**.** For any $S \subseteq A^\mu$, it is easy to check that $\mathbb{I}(S)$ is a congruence on $\mathscr{F}(\mu)$. In view of this, for any subset $S \subseteq A^\mu$ we define

$$\mathscr{I}(S) = \mathscr{F}(\mu)/\mathbb{I}(S)\ .$$

Definition 3.6 (The functor $\mathscr{I}$ on arrows.)**.** Given $S \subseteq A^\mu$ and $T \subseteq A^\nu$, let $\lambda\colon S \to T$ be a definable map, and let $(l_\beta)_{\beta<\nu}$ be a ν-tuple of defining terms for λ. Then there is an induced function

$$\mathscr{I}(\lambda)\colon\ \mathscr{I}(T) \to \mathscr{I}(S)$$

which acts on each $s \in \mathscr{F}(\nu)$ by substitution as follows:

$$\frac{s\big(\,(X_\alpha)_{\beta<\nu}\,\big)}{\mathbb{I}(T)} \in \mathscr{I}(T) \quad \overset{\mathscr{I}(\lambda)}{\longmapsto} \quad \frac{s\big(\,[X_\beta \backslash l_\beta]_{\beta<\nu}\,\big)}{\mathbb{I}(S)} \in \mathscr{I}(S)\ .$$

Definition 3.7 (The functor $\mathscr{V}$ on objects.)**.** By the very definition of $\mathbb{V}$, for any congruence θ on $\mathscr{F}(\mu)$ we have $\mathbb{V}(\theta) \subseteq A^\mu$. We therefore set

$$\mathscr{V}(\mathscr{F}(\mu)/\theta) = \mathbb{V}(\theta)\,.$$

Definition 3.8 (The functor $\mathscr{V}$ on arrows.)**.** Suppose a homomorphism of V-algebras $h\colon\ \mathscr{F}(\mu)/\theta_1 \to \mathscr{F}(\nu)/\theta_2$ is given. For each $\alpha < \mu$, let π_α be the projection term on the α^{th} coordinate, and let π_α/θ_1 denote the equivalence class of π_α modulo θ_1. Fix, for each α, an arbitrary $f_\alpha \in h(\pi_\alpha/\theta_1)$. For any $(p_\beta)_{\beta<\nu} \in \mathbb{V}(\theta_2)$, set

$$\mathscr{V}(h)((p_\beta)_{\beta<\nu}) = \big(\,f_\alpha(\,(p_\beta)_{\beta<\nu}\,)\,\big)_{\alpha<\mu}\,.$$

Let us just notice that the conspicuous issue in both Definitions 3.6 and 3.8 of well-definiteness of the functors $\mathscr{I}$ and $\mathscr{V}$ can be proved to be immaterial.

Once all this is settled, the proof of the following statement becomes routine.

Theorem 3.9. *The functor $\mathscr{V}\colon\ \mathsf{V_p} \longrightarrow \mathsf{G}_{\mathsf{def}}^{\mathrm{op}}$ is left adjoint to the functor $\mathscr{I}\colon \mathsf{G}_{\mathsf{def}}^{\mathrm{op}} \longrightarrow \mathsf{V_p}$. In symbols, $\mathscr{V} \dashv \mathscr{I}$.*

As any adjunction specialises to a categorical equivalence among its fixed points, one becomes naturally interested in possible characterisations of these fixed points. By the definitions of $\mathscr{I}$ and $\mathscr{V}$ and Lemma 3.3 this reduces to studying the closure operators $\mathbb{I} \circ \mathbb{V}$ and $\mathbb{V} \circ \mathbb{I}$.

4 An abstract algebraic version of the *Nullstellensatz*.

Theorem 4.1. *Let A be an algebra in V and θ a congruence of $\mathscr{F}(\mu)$. Then the following are equivalent:*

(i) $\mathbb{I}\,\mathbb{V}(\theta) = \theta$.

(ii) $\theta = \bigcap_{a \in \mathbb{V}(\theta)} \mathbb{I}(a)$.

(iii) $\mathscr{F}(\mu)/\theta$ *is a subdirect product of the family of algebras* $\{\mathscr{F}(\mu)/\,\mathbb{I}(a)\}_{a\in\mathbb{V}(\theta)}$.

The effect of the Theorem above is that there always is a dual equivalence of categories between the full subcategory of V where the objects are subdirect products of algebras of the form $\mathscr{F}(\mu)/\,\mathbb{I}_A(a)$ and the category of "closed" (in the sense of Zariski) subspaces of A^μ, i.e. spaces S for which $S = \mathbb{V}\,\mathbb{I}(S)$.

Corollary 4.2. *For any variety* V *and for any* $A \in \mathsf{V}$ *the following are equivalent:*

(i) *For any* μ *and any completely meet-irreducible congruence* $\theta \subseteq \mathscr{F}(\mu)^2$*, there exists* $a \in A^\mu$ *such that* $\theta = \mathbb{I}_A(a)$*.*

(ii) *For any* μ *and every congruence* $\theta \subseteq \mathscr{F}(\mu)^2$ *one has* $\mathbb{I}\,\mathbb{V}(\theta) = \theta$*.*

In other words, the adjunction given by the pair $\mathscr{I}_A \vdash \mathscr{V}_A$ *is an equivalence itself if and only if* A *satisfies item (i).*

The question whether a certain congruence can be expressed as $\mathbb{I}(a)$ is pivotal in an explicit characterisation of the duality lying behind the adjunction. The following theorem gives a rather effective characterisation of the congruences that can be expressed as such.

Theorem 4.3. *For any congruence* $\theta \subseteq \mathscr{F}(\mu)^2$ *the following are equivalent:*

(i) *there exists some* $a \in A^\mu$ *such that* $\theta = \mathbb{I}_A(a)$*.*

(ii) *the algebra* $\mathscr{F}(\mu)/\theta$ *embeds in* A*.*

5 Applications.

In this section we quickly hint at how a number of classical results can re-proved using the theorems above.

The classical Nullstellensatz. Let k be a field and $k[X_1, ..., X_n]$ be the polynomial ring with coefficients in k. The field k can be regarded as a k-algebra over itself, set V to be the variety of k-algebras generated by k and set $A = k$. The k-algebra $k[X_1, ..., X_n]$ belongs to V. Even more, $k[X_1, ..., X_n]$ is the free algebra over n generators in the variety V. The $\mathbb{V}\,\mathbb{I}$ topology in this case is just the classical Zariski topology, so the above duality here is given by the correspondence between *coordinate algebras* with their morphisms and closed subspaces of k^μ (aka *affine varieties*) and regular maps among them. Upon recalling that homomorphisms of a k-algebra into k have a kernel which is a maximal ideal and that for any ideal I, $\bigcap\{I \subseteq M \mid M \text{ is a maximal ideal}\} = \bigcap\{I \subseteq P \mid P \text{ is a prime ideal}\}$, Theorem 4.3 and 4.1 immediately give Hilbert's classical Nullstellensatz as the characterisation of the congruences fixed by the operator $\mathbb{I}\,\mathbb{V}$. They are exactly *radical congruences*, i.e. those that can be written as the intersection of prime congruences.

Boolean algebras. Let A be the two element Boolean algebra $\{0, 1\}$, then one immediately has that every subdirectly irreducible Boolean algebra embeds into A (A is the *only* subdirectly irreducible Boolean algebra), hence by Theorems 4.3 and 4.1 one gets that there is a dual equivalence between the category of Boolean algebras and closed subspaces of the *generalised* Cantor spaces 2^μ with its standard topology. Finally one notes that these closed sets (with the induced topology) are the most general Boolean spaces (i.e. any other can be realised as such).

Other dualities that appropriately to fit in the above framework are Priestly duality for distributive lattices, the duality between Tychonoff spaces and semisimple MV-algebras, Stone-Gelfand duality for commutative real C^*-algebras, Baker-Beynon duality for finitely presented Riesz spaces and the similar duality for Abelian ℓ-groups.